

A SUFFICIENT CONDITION FOR HAVING BIG PIECES OF BILIPSCHITZ IMAGES OF SUBSETS OF EUCLIDEAN SPACE IN HEISENBERG GROUPS

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ABSTRACT. In this article we extend a euclidean result of David and Semmes to the Heisenberg group by giving a sufficient condition for a k -Ahlfors-regular subset to have big pieces of bilipschitz images of subsets of \mathbb{R}^k . This Carleson type condition measures how well the set can be approximated by the Heisenberg k -planes at different scales and locations. The proof given here follow the paper of David and Semmes.

1. INTRODUCTION

In [5] and [9] Jones and Okikiolu proved that a bounded set $E \subset \mathbb{R}^n$ is contained in a rectifiable curve if and only if

$$(1) \quad \int_0^\infty \int_{\mathbb{R}^n} \beta_\infty^E(x, t)^2 dx \frac{dt}{t^n} < \infty,$$

where

$$\beta_\infty^E(x, t) = \inf_L t^{-1} \sup \left\{ d_e(y, L) : y \in B_E^{d_e}(x, t) \right\},$$

with the infimum taken over all lines in \mathbb{R}^n . Here and in the sequel d_e denotes the euclidean metric in \mathbb{R}^n (for any n in question each time) and $B_Y^\rho(x, r) = B_Y(x, r) = \{y \in X : \rho(y, x) \leq r\}$ for a metric space (X, ρ) , $Y \subset X$, $x \in X$ and $r \geq 0$. In [3] David and Semmes gave a higher dimensional version of the above theorem for k -regular subsets of \mathbb{R}^n (where k is an integer between 0 and n) by showing that a closed k -regular set $E \subset \mathbb{R}^n$ has big pieces of Lipschitz images of \mathbb{R}^k if and only if there is $C < \infty$ such that

$$(2) \quad \int_0^r \int_{B_E(z, r)} \beta_1^E(x, t)^2 d\mathcal{H}_E^k(x) \frac{dt}{t} \leq Cr^k$$

for all $z \in E$ and $r > 0$, where

$$\beta_1^E(x, t) = t^{-k-1} \inf_L \int_{B_E(x, t)} d_e(y, L) d\mathcal{H}_E^k(y)$$

with infimum taken over all k -planes in \mathbb{R}^n . In fact David and Semmes gave in [3] several equivalent conditions to (2) and said that a closed k -regular set $E \subset \mathbb{R}^n$ is uniformly rectifiable if it satisfies these conditions.

Above the metric notions are of course taken with respect to d_e . More generally, we say that a metric space (X, ρ) is k -regular if there exists a constant $C \in \mathbb{R}$ such that $C^{-1}r^k \leq \mathcal{H}_X^k(B_X^\rho(x, r)) \leq Cr^k$ for any $x \in X$ and $r \in]0, \rho(X)[$, where \mathcal{H}_X^k is the k -dimensional Hausdorff measure on (X, ρ) . The smallest such constant C will be denoted by $C_{(X, \rho)}$. Further we say that (X, ρ) has big pieces of bilipschitz images of subsets of \mathbb{R}^k (with constants K and c) if for any $x \in X$ and $r \in]0, \rho(X)[$ there exists a K -bilipschitz function $f : A \rightarrow X$ (w.r.t. the metrics d_e and ρ) with $A \subset B_{\mathbb{R}^k}^{d_e}(0, r)$ such that $\mathcal{H}_X^k(f(A) \cap B_X(x, r)) \geq cr^k$.

In [10] Schul extended the one dimensional result of Jones and Okikiolu to Hilbert spaces (with the condition (1) modified in an appropriate way). Further in [4] Ferrari, Franchi and

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Pajot gave for a compact subset E of the Heisenberg group \mathbb{H}^1 (endowed with its Carnot-Carathéodory metric) an analogue of the condition (1) which measures the deviation of E from a best approximating Heisenberg straight line (i.e. an element of \mathcal{V}^1 , see (5)) at different scales and locations. They showed that this condition is sufficient for E to be contained in a rectifiable curve. Juillet gave in [6] an example which shows that it is not necessary. Following [4] we define for a k -regular subset E of the Heisenberg group \mathbb{H}^n an analogue for (2) and give a proof for the following theorem. For the definitions see Sections 2 and 3. Theorem 1.1 is invariant under bilipschitz change of metric (see also (15)). Note that the Korányi metric d (see (4)) which we below use exclusively is bilipschitz equivalent with the usual Carnot-Carathéodory metric on \mathbb{H}^n .

Theorem 1.1. *Let $k \in \mathbb{N}$ and E be a k -regular subset of the Heisenberg group \mathbb{H}^n . If there is a constant C such that*

$$(3) \quad \int_0^r \int_{B_E(x,r)} \beta_1^E(y,t)^2 d\mathcal{H}_E^k(y) \frac{dt}{t} \leq Cr^k \quad \text{for all } x \in E \text{ and } r > 0,$$

then E has big pieces of bilipschitz images of subsets of \mathbb{R}^k .

The proof given here follows [3]. A similar method is applied also in [7]. For readability and consistency we give a quite detailed proof although mostly the adaptation from [3] is trivial or at least straightforward.

In this article $|\cdot|$ denotes the euclidean k -norm for any k in question. The cardinality of a finite set X is denoted by $\#X$. Further $\mathcal{P}(X) = \{Y : Y \subset X\}$ for any set X , and the symbol \bar{f} is used to denote an average integral.

2. SOME NOTATIONS AND PRELIMINARIES ON HEISENBERG GROUPS

The Heisenberg group \mathbb{H}^n is the unique simply connected and connected Lie group of step two and dimension $2n + 1$ with one dimensional center. As a set it may be identified with \mathbb{R}^{2n+1} . The points $x \in \mathbb{H}^n$ are written as $x = (x', x_{2n+1})$ with $x' \in \mathbb{R}^{2n}$ and $x_{2n+1} \in \mathbb{R}$. The group operation is given by

$$x \cdot y = (x' + y', x_{2n+1} + y_{2n+1} + 2A(x', y')),$$

where

$$A(x', y') = \sum_{i=1}^n (x_{i+n} y_i - x_i y_{i+n}).$$

Note that the inverse of x , denoted also by x^{-1} , is $-x = (-x', -x_{2n+1})$ and the neutral element is $(0, 0)$. We equip \mathbb{H}^n by a metric d defined by

$$(4) \quad d(x, y) = \|y^{-1} \cdot x\|, \quad \text{where } \|x\| = (|x'|^4 + x_{2n+1}^2)^{1/4}.$$

The metric d is left invariant i.e. for each $p \in \mathbb{H}^n$ the left translation $\tau_p : x \mapsto p \cdot x$ is an isometry from (\mathbb{H}^n, d) to itself. Note that for $x, y \in \mathbb{R}^{2n} \times \{0\}$ the conditions $A(x', y') = 0$, $x \cdot y = x + y$ and $d(x, y) = |x - y|$ are equivalent.

A linear subspace $V \subset \mathbb{R}^{2n}$ is said to be *isotropic* if $A(x, y) = 0$ for all $x, y \in V$. For $k \in \mathbb{N}$ denote

$$\mathcal{V}_0^k = \{ V \times \{0\} : V \text{ is an isotropic } k\text{-dimensional linear subspace of } \mathbb{R}^{2n} \}.$$

In other words \mathcal{V}_0^k is the collection of the k -dimensional homogenous horizontal subgroups of \mathbb{H}^n (see [1]). We note that $\mathcal{V}_0^k \neq \emptyset$ if and only if $0 \leq k \leq n$. Set

$$(5) \quad \mathcal{V}^k = \{ \tau_p(V) : V \in \mathcal{V}_0^k \text{ and } p \in \mathbb{H}^n \}.$$

Each $V \in \mathcal{V}^k$ is a k -dimensional affine subspace of \mathbb{R}^{2n+1} because τ_p is an affine mapping whose linear part has determinant 1. Note also that for any $V \in \mathcal{V}^k$

$$(6) \quad d(x, y) = |x' - y'| \quad \text{for all } x, y \in V.$$

Namely, let $V = \tau_p(V_0)$ for $V_0 \in \mathcal{V}_0^k$, $x = p \cdot x_0$ and $y = p \cdot y_0$. Since $(p \cdot z)' = p' + z'$ for any $z \in \mathbb{H}^n$, one has $d(x, y) = d(x_0, y_0) = |x'_0 - y'_0| = |x' - y'|$.

For $V \in \mathcal{V}^k$ we define the projection $P_V : \mathbb{H}^n \rightarrow V$ by setting

$$P_V = \tau_p \circ P_{\tau_{-p}(V)}^e \circ \tau_{-p},$$

where $p \in V$ and $P_L^e : \mathbb{H}^n \rightarrow L$ is the euclidean orthogonal projection to the linear subspace $L \in \mathcal{V}_0^k$ (called *horizontal projection* in [8]). Note that the definition of P_V is correct, because letting $V_0 \in \mathcal{V}_0^k$ and $x \in \mathbb{H}^n$

$$P_{V_0}^e(a \cdot x) = P_{V_0}^e(a' + x', 0) = P_{V_0}^e(a', 0) + P_{V_0}^e(x', 0) = P_{V_0}^e(a) \cdot P_{V_0}^e(x)$$

for every $a \in \mathbb{H}^n$, and hence

$$p \cdot v \cdot P_{V_0}^e((p \cdot v)^{-1} \cdot x) = p \cdot v \cdot (P_{V_0}^e(-v) \cdot P_{V_0}^e(p^{-1} \cdot x)) = p \cdot P_{V_0}^e(p^{-1} \cdot x).$$

for any $p \in \mathbb{H}^n$ and $v \in V_0$. It is easy to see (using the left invariance of d) that P_V is 1-Lipschitz for any $V \in \mathcal{V}^k$.

Remark 2.1. A linear map $\varphi : \mathbb{H}^n \rightarrow \mathbb{H}^n$ is called a *rotation* if $\varphi(x)_{2n+1} = x_{2n+1}$, $A(x', y') = A(\varphi(x)', \varphi(y)')$ and $|\varphi(x)' - \varphi(y)'| = |x' - y'|$ for all x and y . If φ is a rotation then clearly $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$ and $\varphi(V) \in \mathcal{V}^k$ for any for any $x, y \in \mathbb{H}^n$ and $V \in \mathcal{V}^k$. Hence

$$P_{\tau_p \circ \varphi(V)} \circ \tau_p \circ \varphi = \tau_p \circ \varphi \circ P_V$$

for any $p \in \mathbb{H}^n$, $V \in \mathcal{V}^k$ and rotation φ . Namely, if $V = \tau_q(V_0)$ for $q \in \mathbb{H}^n$ and $V_0 \in \mathcal{V}_0^k$ then for $x \in \mathbb{H}^n$

$$\begin{aligned} P_{\tau_p \circ \varphi(V)}(p \cdot \varphi(x)) &= P_{p \cdot \varphi(q) \cdot \varphi(V_0)}(p \cdot \varphi(x)) \\ &= p \cdot \varphi(q) \cdot P_{\varphi(V_0)}^e((p \cdot \varphi(q))^{-1} \cdot p \cdot \varphi(x)) = p \cdot \varphi(q) \cdot P_{\varphi(V_0)}^e(\varphi(x) - \varphi(q)) \\ &= p \cdot \varphi(q) \cdot \varphi(P_{V_0}^e(x - q)) = p \cdot \varphi(q \cdot P_{V_0}^e(x - q)) = p \cdot \varphi(P_V(x)). \end{aligned}$$

For any $V_0, W_0 \in \mathcal{V}_0^k$ there is a rotation φ such that $W_0 = \varphi(V_0)$ (see [1]). Hence for any $V, W \in \mathcal{V}^k$ there is a rotation φ and $p \in \mathbb{H}^n$ such that $W = \tau_p \circ \varphi(V)$. Notice that the rotations are isometries.

Denote $X_k = \{x \in \mathbb{H}^n : x_i = 0 \text{ for all } i > k\}$.

Lemma 2.2. For any $x \in \mathbb{H}^n$ and $V \in \mathcal{V}^k$

$$d(x, P_V(x)) \leq 3d(x, V).$$

Proof. By the left invariance of d this follows from [8] (at least with 3 replaced by some constant). Let us give here another proof by a direct calculation. By 2.1 one only needs show that $d(x, P_{X_k}(x)) \leq 3d(x, X_k)$ for all $x \in \mathbb{H}^n$. Let $(z, y) \in \mathbb{R}^k \times \mathbb{R}^{2n-k}$, $u \in X_k$ and $C > 1$. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by setting $f(t) = Cd((z, y, t), u)^4 - d((z, y, t), P_{X_k}(z, y, t))^4$. Let $t \in \mathbb{R}$. By denoting $T = -2 \sum_{i=1}^k u_i y_{n-k+i}$ and $U = -2 \sum_{i=1}^k z_i y_{n-k+i}$ we have

$$f(t) = C(|z - u|^2 + |y|^2)^2 + C(t - T)^2 - |y|^4 - (t - U)^2.$$

Now

$$f(t) \geq f\left(\frac{CT - U}{C - 1}\right) = C(|z - u|^2 + |y|^2)^2 - \frac{C(T - U)^2}{C - 1} - |y|^4.$$

Since

$$|T - U| = 2 \left| \sum_{i=1}^k (z_i - u_i) y_{n-k+i} \right| \leq 2|z - u||y| \leq (|z - u|^2 + |y|^2),$$

we have

$$f(t) \geq \frac{C(C - 2)}{C - 1} (|z - u|^2 + |y|^2)^2 - |y|^4 \geq 0$$

by choosing $C \geq 3$. □

For $V, W \in \mathcal{V}^k$ denote

$$\angle(V, W) = \min\{C \geq 1 : d(x, y) \leq Cd(P_W(x), P_W(y)) \text{ for all } x, y \in V\}.$$

Let P_L^e denote also the euclidean orthogonal projection from \mathbb{R}^{2n} to an affine subspace $L \subset \mathbb{R}^{2n}$. For any $Y \subset \mathbb{H}^n$ we write $Y' = \{x' : x \in Y\}$.

Let $V = \tau_p(V_0)$ for $p \in \mathbb{H}^n$ and $V_0 \in \mathcal{V}_0^k$. Then and

$$P_V(x)' = (p \cdot P_{V_0}^e(x - p))' = p' + P_{V_0}^e(x - p)' = p' - P_{V_0}^e(p)' + P_{V_0}^e(x)'$$

for any x (by the linearity of $P_{V_0}^e$). Thus, since $V' = p' + V_0'$, we have by (6)

$$(7) \quad d(P_V(x), P_V(y)) = |P_V(x)' - P_V(y)'| = |P_{V_0}^e(x)' - P_{V_0}^e(y)'| = |P_{V'}^e(x') - P_{V'}^e(y')|$$

for any $x, y \in \mathbb{H}^n$. Particularly the following equality holds.

Lemma 2.3. Let $V, W \in \mathcal{V}^k$. Then

$$\angle(V, W) = \min\{C \geq 1 : |x - y| \leq C|P_{W'}^e(x) - P_{W'}^e(y)| \text{ for all } x, y \in V'\}.$$

3. BETA NUMBERS AND DYADIC CUBES

Let $k \in \{1, \dots, n\}$. From this on we assume that E is a k -regular subset of \mathbb{H}^n . Denote $B(x, r) = B_{\mathbb{H}^n}^d(x, r)$ and $\mu = \mathcal{H}^k|_E$, where \mathcal{H}^k is the k -dimensional Hausdorff measure on \mathbb{H}^n (with respect the metric d). By [2] there exist constants $\alpha, D \in]1, \infty[$ (depending only on k and the regularity constant C_E) and a collection $\Delta^* = \bigcup_{j \in \mathbb{Z}} \Delta_j \subset \mathcal{P}(E)$ such that each $Q \in \Delta^*$ is open in E and

$$(8) \quad \mu\left(E \setminus \bigcup_{Q \in \Delta_j} Q\right) = 0 \text{ for all } j \in \mathbb{Z}.$$

$$(9) \quad \text{If } Q, R \in \Delta_j \text{ and } Q \neq R, \text{ then } Q \cap R = \emptyset.$$

$$(10) \quad \text{If } Q \in \Delta_j, R \in \Delta_l \text{ and } j \leq l, \text{ then } Q \subset R \text{ or } Q \cap R = \emptyset.$$

$$(11) \quad d(Q) \leq D\alpha^j \text{ for all } Q \in \Delta_j.$$

$$(12) \quad \text{If } Q \in \Delta_j, \text{ then } B(x, D^{-1}\alpha^j) \cap E \subset Q \text{ for some } x \in E.$$

$$(13) \quad \mu(\{x \in Q : d(x, E \setminus Q) \leq t\alpha^j\}) \leq Dt^{1/D}\mu(Q) \text{ for all } Q \in \Delta_j, t > 0.$$

By (11) and (12) also $D^{-k}C_E^{-3}\alpha^{jk} \leq \mu(Q) \leq C_ED^k\alpha^{jk}$ for $Q \in \Delta_j$ if $\alpha^j \leq Dd(E)$. Thus by defining $J_0 = \inf\{j : \Delta_j = \{E\}\}$ (here $J_0 = \infty$ if $d(E) = \infty$) and $\mathcal{Z} = \{j \in \mathbb{Z} : j \leq J_0\}$ and taking D larger we can assume that

$$(14) \quad D^{-1}\alpha^j \leq d(Q) \leq D\alpha^j \quad \text{and} \quad D^{-1}\alpha^{jk} \leq \mu(Q) \leq D\alpha^{jk}$$

for all $Q \in \Delta_j, j \in \mathcal{Z}$. Set $\Delta = \bigcup_{j \in \mathcal{Z}} \Delta_j$.

If $(X, \rho) \in \{(\mathbb{H}^n, d), (\mathbb{R}^k, d_e)\}$ we write $Z(r) = \{x \in X : \rho(x, Z) \leq r\}$ for any $Z \subset X$ and $r > 0$. We further denote

$$\begin{aligned} \lambda Q &= Q((\lambda - 1)d(Q)) \cap E, \\ \lambda F &= Q((\lambda - 1)d_e(F)) \end{aligned}$$

for any $Q \subset E, F \subset \mathbb{R}^k$ and $\lambda > 1$. Each constant in this article may depend on k and C_E without special mention. For future let ε and δ be small positive constants and K_0 and K large constants. We will fix K_0 first, δ second and ε after K . The constants C in Sections 3–7 depend on ε, K, δ or K_0 only if it is separately mentioned. Eventually every constant will depend only on k, C_E and C from (3).

For $x \in \mathbb{H}^n$, $t > 0$ and $F \subset E$ with $d(F) > 0$ denote

$$(15) \quad \begin{aligned} \beta_1(x, t) &= \beta_1^E(x, t) = t^{-k-1} \inf_{V \in \mathcal{V}^k} \int_{B(x, t)} d(y, V) d\mu y, \\ \beta_\infty(F) &= d(F)^{-1} \inf_{V \in \mathcal{V}^k} \sup \{ d(y, V) : y \in F \}. \end{aligned}$$

We say that E satisfies the *weak geometric lemma* if for each $\lambda_1 > 0$ and $\lambda_2 > 1$ there is a constant $C(\lambda_1, \lambda_2)$ such that

$$(16) \quad \sum_{j \in \mathcal{Z}} \sum_{\substack{Q \in \Delta_j \\ Q \subset R \\ \beta_\infty(\lambda_2 Q) > \lambda_1}} \mu(Q) \leq C(\lambda_1, \lambda_2) \mu(R) \quad \text{for all } R \in \Delta.$$

Denote

$$\mathcal{G}_1 = \left\{ Q \in \Delta : KQ \subset V(\varepsilon^2 d(Q)) \text{ for some } V \in \mathcal{V}^k \right\}.$$

Clearly (16) implies

$$(17) \quad \sum_{j \in \mathcal{Z}} \sum_{\substack{Q \in \Delta_j \setminus \mathcal{G}_1 \\ Q \subset R}} \mu(Q) \leq C(\varepsilon, K) \mu(R) \quad \text{for all } R \in \Delta.$$

Note also that (3) implies (16) (and hence (17)). For the proof see for example [3]. (This clearly remains valid even if \mathbb{H}^n and \mathcal{V}^k are replaced by any k -regular metric space (X, ρ) and $\mathcal{A} \subset \mathcal{P}(X)$ with $\inf\{\rho(x, V) : V \in \mathcal{A}\} = 0$ for all $x \in E$.) For each $Q \in \mathcal{G}_1$ we let $V_Q \in \mathcal{V}^k$ be such that $KQ \subset V_Q(\varepsilon^2 d(Q))$.

Lemma 3.1. *There is a constant $c > 0$ such that for any $Q \in \mathcal{G}_1$ there exists $\{y_0, \dots, y_k\} \subset V_Q \cap Q(\varepsilon^2 d(Q))$ such that $d(y_{i+1}, L_i) > cd(Q)$ for all $i \in \{0, \dots, k-1\}$, where $L_i \subset V_Q$ is the i -dimensional affine subspace with $\{y_0, \dots, y_i\} \subset L_i$.*

Proof. Choose some $x_0 \in Q$ and take $L_0 = \{y_0\} \subset V_Q$ such that $d(x_0, y_0) = d(x_0, V_Q) \leq \varepsilon^2 d(Q)$. Assume now that $i < k$ and the i -dimensional affine subspace $L_i \subset V_Q$ is defined. Suppose to the contrary that $V_Q \cap Q(\varepsilon^2 d(Q)) \subset L_i(cd(Q))$, where $c > 0$ is a constant determined later. Then $Q \subset L_i((c + \varepsilon)d(Q))$. Denote $F = L_i \cap Q((c + \varepsilon)d(Q))$ and let H be a maximal subset of F such that $d(z, w) > ad(Q)$ for distinct $z, w \in H$, where $a > 0$ is a constant fixed later. Then

$$Q \subset F((c + \varepsilon)d(Q)) \subset \bigcup_{y \in H} B(y, (c + \varepsilon + a)d(Q))$$

and

$$\#H \cdot \left(\frac{ad(Q)}{2} \right)^i \leq (1 + 2(c + \varepsilon))^i d(Q)^i$$

by (6). Picking $x(y) \in B(y, (c + \varepsilon + a)d(Q))$ for each $y \in H$ we get

$$\begin{aligned} \mu(Q) &\leq \sum_{y \in H} \mu(B(x(y), 2(c + \varepsilon + a)d(Q))) \leq \#H \cdot C_E 2^k (c + \varepsilon + a)^k d(Q)^k \\ &\leq \left(\frac{2 + 4(c + \varepsilon)}{a} \right)^i C_E 2^k (c + \varepsilon + a)^k d(Q)^k. \end{aligned}$$

By choosing a and c suitably (depending only on k and C_E) and then $\varepsilon > 0$ small enough one gets the contradiction with (14). \square

Lemma 3.2. *If $Q \in \mathcal{G}_1$ and $V \in \mathcal{V}^k$ is such that $Q \subset V(2K\varepsilon^2 d(Q))$, then $\angle(V_Q, V) \leq 1 + \varepsilon$.*

Proof. Let $\{y_0, \dots, y_k\} \subset V_Q \cap Q(\varepsilon^2 d(Q))$ as in Lemma 3.1. Then $\{y_0, \dots, y_k\} \subset V((1 + 2K)\varepsilon^2 d(Q))$ and $d(y_i, y_j) > cd(Q)$ for all distinct $i, j \in \{0, \dots, k\}$. Thus by Lemma 2.2

$$\begin{aligned} d(y_i, y_j) &\leq d(P_V(y_i), P_V(y_j)) + d(y_i, P_V(y_i)) + d(y_j, P_V(y_j)) \\ &\leq d(P_V(y_i), P_V(y_j)) + 6(1 + 2K)\varepsilon^2 d(Q) \\ &\leq d(P_V(y_i), P_V(y_j)) + 6(1 + 2K)c^{-1}\varepsilon^2 d(y_i, y_j) \end{aligned}$$

for all $i, j \in \{1, \dots, k\}$, where c is as in Lemma 3.1. Choosing ε small enough depending on K (and c) we get by (7) and (6) that $|x - y| \leq (1 + \varepsilon)|P_{V'}^e(x) - P_{V'}^e(y)|$ for all $x, y \in V'_Q$. The claim now follows from Lemma 2.3. \square

4. STOPPING TIME REGIONS

In this section we mostly follow [3, Sections 7 and 8]. We use the same assumptions and notations as in Section 3 assuming additionally that E satisfies the weak geometric lemma (16). For any $Q \in \Delta^*$ denote $\mathcal{C}(Q) = \{R \in \Delta_{j_Q-1} : R \subset Q\}$, where $j_Q = \min\{j \in \mathbb{Z} : Q \in \Delta_j\}$. If $j_Q < J_0$, we also denote by $O(Q)$ the unique $R \in \Delta^*$ for which $Q \in \mathcal{C}(R)$. For any $\mathcal{S} \subset \Delta$ let $\min(\mathcal{S})$ be the set of minimal (with respect to inclusion) cubes in \mathcal{S} .

Lemma 4.1. *There is $\mathcal{G} \subset \mathcal{G}_1$ and $\mathfrak{F} \subset \mathcal{P}(\mathcal{G})$ such that $\mathcal{G} = \bigcup_{\mathcal{S} \in \mathfrak{F}} \mathcal{S}$ and the following conditions are satisfied:*

(F1) *For all $R \in \Delta$*

$$\sum_{j \in \mathbb{Z}} \sum_{\substack{Q \in \Delta_j \setminus \mathcal{G} \\ Q \subset R}} \mu(Q) \leq C(\varepsilon, K)\mu(R).$$

(F2) *If $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{F}$ and $\mathcal{S}_1 \neq \mathcal{S}_2$, then $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$.*

(F3) *Each $\mathcal{S} \in \mathfrak{F}$ has a largest element with respect to inclusion, denoted by $Q(\mathcal{S})$.*

(F4) *If $Q \in \mathcal{S}$, $R \in \Delta$ and $Q \subset R \subset Q(\mathcal{S})$, then $R \in \mathcal{S}$.*

(F5) *$\angle(V_Q, V_{Q(\mathcal{S})}) \leq 1 + \delta$ for all $Q \in \mathcal{S}$.*

(F6) *If $Q \in \mathcal{S}$, $\mathcal{C}(Q) \subset \mathcal{G}$ and $\angle(V_R, V_{Q(\mathcal{S})}) \leq 1 + \delta$ for all $R \in \mathcal{C}(Q)$, then $\mathcal{C}(Q) \subset \mathcal{S}$.*

(F7) *$Q \in \min(\mathcal{S})$ if and only if the following two conditions are satisfied:*

- $Q \in \mathcal{S}$
- $\mathcal{C}(Q) \setminus \mathcal{G} \neq \emptyset$ or $\angle(V_R, V_{Q(\mathcal{S})}) > 1 + \delta$ for some $R \in \mathcal{C}(Q)$

Proof. Assume first that E is unbounded. Let $p \in E$ and set $\mathcal{D} = \min(\bigcup_{j \in \mathbb{N}} \mathcal{D}_j)$, where $\mathcal{D}_j = \{R \in \Delta_j : B(p, \alpha^j) \cap O(R) \neq \emptyset\}$. Clearly each $Q \in \Delta$ is included in some $R \in \bigcup_{j \in \mathbb{N}} \mathcal{D}_j$. Let $j \in \mathbb{Z}$ and $Q \in \Delta_j$ be such that $d(p, Q) > (D^3\alpha + 1)\alpha^j$. We next show that there exists $R \in \mathcal{D}$ such that $Q \subset R$. We just let R be the minimal cube in $\bigcup_{i \in \mathbb{N}} \mathcal{D}_i$ such that $Q \subset R$. Since trivially $\mathcal{D}_0 \subset \mathcal{D}$, we assume $R \in \mathcal{D}_i$ for $i \geq 1$. Now $i > j$ because $Q \notin \mathcal{D}_j$. Thus $Q \subset R^*$ for some $R^* \in \mathcal{C}(R)$. By the minimality of R we have $B(p, \alpha^{j_{R^*}-1}) \cap R = \emptyset$ from which we conclude $R \in \mathcal{D}$. So by the regularity there is a constant C such that for every $j \in \mathbb{Z}$ there is at most C cubes in Δ_j which are not contained in any cube in \mathcal{D} . If E is bounded we set $\mathcal{D} = \Delta_{J_0}$ (which contains only one cube).

Defining $\mathcal{G} = \mathcal{G}_1 \setminus \{Q \in \Delta : Q \not\subset R \text{ for all } R \in \mathcal{D}\}$ the condition (F1) holds by (17) and the previous discussion. For each $R \in \mathcal{D}$ we partition $\mathcal{G}(R) = \{Q \in \mathcal{G} : Q \subset R\}$ into a family of "stopping time regions" as follows: Let Q_0 be a maximal element in $\mathcal{G}(R)$. The family \mathcal{S} is defined to be the unique subset of $\mathcal{G}(R)$ whose largest element is Q_0 and which satisfies the conditions (F3)–(F7). Then we repeat the process for $\mathcal{G}(R) \setminus \mathcal{S}$. Since $R_1 \cap R_2 = \emptyset$ for distinct $R_1, R_2 \in \mathcal{D}$, the condition (F2) is satisfied. \square

Notice that (F6) and (F7) imply

$$(18) \quad Q \in \mathcal{S} \setminus \min(\mathcal{S}) \implies \mathcal{C}(Q) \subset \mathcal{S}.$$

For $\mathcal{S} \in \mathfrak{F}$ denote

$$\begin{aligned} m_1(\mathcal{S}) &= \{Q \in \min(\mathcal{S}) : \mathcal{C}(Q) \setminus \mathcal{G} \neq \emptyset\}, \\ m_2(\mathcal{S}) &= \min(\mathcal{S}) \setminus m_1(\mathcal{S}) \end{aligned}$$

and further

$$\begin{aligned} \mathfrak{F}_1 &= \left\{ \mathcal{S} \in \mathfrak{F} : \mu\left(\bigcup_{Q \in m_1(\mathcal{S})} Q\right) \geq \mu(Q(\mathcal{S}))/4 \right\}, \\ \mathfrak{F}_2 &= \left\{ \mathcal{S} \in \mathfrak{F} : \mu\left(Q(\mathcal{S}) \setminus \bigcup_{Q \in \min(\mathcal{S})} Q\right) \geq \mu(Q(\mathcal{S}))/4 \right\}, \\ \mathfrak{F}_3 &= \left\{ \mathcal{S} \in \mathfrak{F} : \mu\left(\bigcup_{Q \in m_2(\mathcal{S})} Q\right) \geq \mu(Q(\mathcal{S}))/2 \right\}. \end{aligned}$$

Clearly $\mathfrak{F} = \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_3$.

Lemma 4.2. *There is a constant $C = C(\varepsilon, K)$ such that*

$$\sum_{\substack{\mathcal{S} \in \mathfrak{F}_1 \cup \mathfrak{F}_2 \\ Q(\mathcal{S}) \subset R}} \mu(Q(\mathcal{S})) \leq C\mu(R) \quad \text{for all } R \in \Delta.$$

Proof. Let $R \in \Delta$. By (14) and (F1)

$$\sum_{\substack{\mathcal{S} \in \mathfrak{F}_1 \\ Q(\mathcal{S}) \subset R}} \mu(Q(\mathcal{S})) \leq 4 \sum_{\substack{\mathcal{S} \in \mathfrak{F}_1 \\ Q(\mathcal{S}) \subset R}} \mu\left(\bigcup_{Q \in m_1(\mathcal{S})} Q\right) \leq 4D^2\alpha^k \sum_{\substack{Q \in \Delta \setminus \mathcal{G} \\ Q \subset R}} \mu(Q) \leq 4D^2\alpha^k C(\varepsilon, K)\mu(R).$$

Let $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{F}$, $\mathcal{S}_1 \neq \mathcal{S}_2$. Then $Q(\mathcal{S}_1) \neq Q(\mathcal{S}_2)$ by (F2). If $Q(\mathcal{S}_1) \cap Q(\mathcal{S}_2) \neq \emptyset$ then (10) implies $Q(\mathcal{S}_1) \subset Q(\mathcal{S}_2)$ or $Q(\mathcal{S}_2) \subset Q(\mathcal{S}_1)$. Assume that $Q(\mathcal{S}_1) \subset Q(\mathcal{S}_2)$ and take minimal $Q \in \mathcal{S}_2$ such that $Q(\mathcal{S}_1) \subset Q$. Since $Q \neq Q(\mathcal{S}_1)$ by (F2) one has $Q \in \min(\mathcal{S}_2)$ by (18). Thus the sets $Q(\mathcal{S}) \setminus \bigcup_{Q \in \min(\mathcal{S})} Q$, $\mathcal{S} \in \mathfrak{F}$, are disjoint and

$$\sum_{\substack{\mathcal{S} \in \mathfrak{F}_2 \\ Q(\mathcal{S}) \subset R}} \mu(Q(\mathcal{S})) \leq 4 \sum_{\substack{\mathcal{S} \in \mathfrak{F}_2 \\ Q(\mathcal{S}) \subset R}} \mu\left(Q(\mathcal{S}) \setminus \bigcup_{Q \in \min(\mathcal{S})} Q\right) \leq 4\mu(R).$$

□

Now the goal is to show that

$$(19) \quad \sum_{\substack{\mathcal{S} \in \mathfrak{F}_3 \\ Q(\mathcal{S}) \subset R}} \mu(Q(\mathcal{S})) \leq C\mu(R) \quad \text{for all } R \in \Delta.$$

The full assumption (3) will be used (instead of the weaker condition (16)) only on page 17 to get (19). After this Theorem 1.1 follows quite easily by the following lemma (see Section 8).

For any $\mathcal{S} \in \mathfrak{F}$ define the function $h_{\mathcal{S}} : \mathbb{H}^n \rightarrow \mathbb{R}$ by setting

$$h_{\mathcal{S}}(x) = \inf\{d(x, Q) + d(Q) : Q \in \mathcal{S}\}.$$

Lemma 4.3. *If $\mathcal{S} \in \mathfrak{F}$ and $x, y \in K_0Q(\mathcal{S})$ with $d(x, y) > D^{-2} \min\{h_{\mathcal{S}}(x), h_{\mathcal{S}}(y)\}$, then $d(x, y) \leq (1 + 2\delta)d(P_{V_{Q(\mathcal{S})}}(x), P_{V_{Q(\mathcal{S})}}(y))$.*

Proof. Assume that $d(x, y) > D^{-2}h_{\mathcal{S}}(x)$ and choose $Q \in \mathcal{S}$ such that

$$d(x, y) > D^{-2}(d(x, Q) + d(Q)).$$

Let $R \in \mathcal{S}$ be the minimal cube such that $Q \subset R$ and $d(K_0R) \geq d(x, y)$. Then

$$d(y, R) \leq d(x, y) + d(x, R) \leq d(x, y) + d(x, Q) < (1 + D^2)d(x, y)$$

and $d(R) \leq D^2(1 + \alpha)d(x, y)$ by (14). Let $z, w \in V_R$ with $d(x, z) = d(x, V_R)$ and $d(y, w) = d(y, V_R)$. Choosing K large enough (depending on K_0 and D) and denoting $P = P_{V_{Q(S)}}$ one gets by (F5)

$$\begin{aligned} d(P(x), P(y)) &\geq d(P(z), P(w)) - d(P(x), P(z)) - d(P(y), P(w)) \\ &\geq d(P(z), P(w)) - d(x, z) - d(y, w) \geq (1 + \delta)^{-1}d(z, w) - 2\varepsilon d(R) \\ &\geq (1 + \delta)^{-1}(d(x, y) - 2\varepsilon d(R)) - 2\varepsilon d(R) > ((1 + \delta)^{-1} - 4D^2\varepsilon(1 + \alpha))d(x, y). \end{aligned}$$

The claim now follows by choosing ε small enough (depending on δ). \square

5. FUNCTION g FOR \mathcal{S}

In this section we follow [3, Section 8] and use the same assumptions and notations as in Section 4. Let $\mathcal{S} \in \mathfrak{F}$ be fixed and assume (in order to simplify notations) that $V_{Q(\mathcal{S})} = X_k$. Then $P_{V_{Q(\mathcal{S})}} = P_{X_k}^e$ and $d(p, q) = |p - q|$ for any $p, q \in X_k$. Because of the latter fact it is natural to denote by $d(F)$ the euclidean diameter of F and by $d(p, F)$ the euclidean distance of p and F for any $F \subset \mathbb{R}^k$ and $p \in \mathbb{R}^k$ (though d is a metric in \mathbb{H}^n). We write $P(x) = (x_1, \dots, x_k)$ and $P^\perp(x) = (x_{k+1}, \dots, x_{2n})$ for any $x \in \mathbb{H}^n$. Denote also $B^k(p, r) = \{q \in \mathbb{R}^k : |q - p| \leq r\}$ for $p \in \mathbb{R}^k$ and $r \geq 0$. The letter C in the calculations in Sections 5–8 denotes always some constant but distinct appearances do not necessarily refer to the same constant (even if they are in the same inequality chain).

Define the function $H : \mathbb{R}^k \rightarrow \mathbb{R}$ by setting

$$H(p) = \inf\{h(x) : P(x) = p\}$$

and set $Z = \{x \in E : h(x) = 0\}$. Here we write shortly $h = h_{\mathcal{S}}$. We immediately see that

$$(20) \quad H(p) = \inf\{d(p, P(Q)) + d(Q) : Q \in \mathcal{S}\}.$$

for any $p \in \mathbb{R}^k$. Namely, the inequality $H(p) \geq \inf\{d(p, P(Q)) + d(Q) : Q \in \mathcal{S}\}$ follows from the 1-Lipschitzness of P . The opposite inequality holds because for any $y \in \mathbb{H}^n$ and $p \in \mathbb{R}^k$ one can obviously choose $x \in P^{-1}(\{p\})$ such that $d(x, y) = d(p, P(y))$. Note that $H(p) = 0$ if and only if $p \in P(Z)$ (for example by the Bolzano–Weierstrass theorem).

For each $p \in \mathbb{R}^k \setminus P(Z)$ let R_p be the largest dyadic cube in \mathbb{R}^k containing p and satisfying

$$(21) \quad 20d(R_p) \leq \inf\{H(u) : u \in R_p\}.$$

Such a cube R_p exists, because $H(p) > 0$ and H is continuous (1-Lipschitz). Let $\{R_i : i \in I\} \subset \{R_p : p \in \mathbb{R}^k \setminus P(Z)\}$ be such that $\{R_i\}_{i \in I}$ covers $\mathbb{R}^k \setminus P(Z)$ and $\text{int } R_i \cap \text{int } R_j = \emptyset$ for distinct $i, j \in I$. Notice that I is countable and $R_i \cap P(Z) = \emptyset$ for any $i \in I$.

By the definition (21) and the 1-Lipschitzness of H

$$(22) \quad 10d(R_i) \leq H(p) \leq 60d(R_i) \quad \text{for any } p \in 10R_i, i \in I.$$

This gives the following lemma.

Lemma 5.1. *There is a constant C such that whenever $10R_i \cap 10R_j \neq \emptyset$ for $i, j \in I$ then $C^{-1}d(R_j) \leq d(R_i) \leq Cd(R_j)$.*

Let $x_0 \in Q(\mathcal{S})$ be any fixed point. Denote $U_j = B^k(P(x_0), 2^{-j}K_0d(Q(\mathcal{S})))$ and $I_j = \{i \in I : R_i \cap U_j \neq \emptyset\}$ for $j \in \mathbb{R}$. By (20) and (22) there exist constants C_0 (which may depend on K_0 by $C_0 = CK_0$) and C such that for each $i \in I_0$ there is $Q_i \in \mathcal{S}$ for which

$$(23) \quad C_0^{-1}d(R_i) \leq d(Q_i) \leq Cd(R_i),$$

$$(24) \quad d(P(Q_i) \cup R_i) \leq Cd(R_i).$$

(In (24) we use the fact that P is 1-Lipschitz.)

For each $i \in I_0$ let $A_i : \mathbb{R}^k \rightarrow \mathbb{R}^{2n-k}$ be the affine function whose graph is V'_{Q_i} . By Lemma 2.3 and (F5) (and by choosing $\delta \leq 1$)

$$(25) \quad \text{Lip}(A_i) \leq \sqrt{(1+\delta)^2 - 1} < 2\sqrt{\delta}.$$

Lemma 5.2. *There is a constant C such that whenever $10R_i \cap 10R_j \neq \emptyset$ for $i, j \in I_0$ then $d(Q_i \cup Q_j) \leq Cd(R_j)$ and*

$$|A_i(p) - A_j(p)| \leq C\sqrt{\varepsilon}d(R_j) \quad \text{for all } p \in 100R_j.$$

Proof. For the first part let $x, y \in Q_i \cup Q_j$. By (23) and Lemma 5.1 one may assume that $x \in Q_i$ and $y \in Q_j$ are such that $d(x, y) \geq d(Q_j)$. Since by definition $h(y) \leq d(Q_j)$, Lemma 4.3, (24) and Lemma 5.1 give

$$d(x, y) \leq (1 + 2\delta)|P(x) - P(y)| \leq Cd(R_j).$$

Thus by choosing K large enough depending on K_0 (23) gives $Q_i \subset KQ_j$ (and $Q_j \subset KQ_i$). Now Lemma 3.2 (with $Q = Q_j$ and $V = V_{Q_i}$) gives

$$\angle(V_{Q_j}, V_{Q_i}) \leq 1 + \varepsilon.$$

Let $p \in 100R_j$ and $z \in Q_j$. Take $y \in V_{Q_j}$ such that $d(y, z) \leq \varepsilon d(Q_j)$. Then by (23) and Lemma 5.1

$$|y' - P_{V'_{Q_i}}^e(y')| = d_e(y', V'_{Q_i}) \leq d(y, z) + d(z, V_{Q_i}) \leq \varepsilon(d(Q_i) + d(Q_j)) \leq C\varepsilon d(R_j)$$

and by (23) and (24)

$$|P(y) - p| \leq |P(y) - P(z)| + |P(z) - p| \leq d(y, z) + d(P(Q_j) \cup 100R_j) \leq Cd(R_j).$$

Denote $v = (p, A_j(p))$. Using the Pythagorean theorem, the above estimates, Lemma 2.3 and (F5)

$$\begin{aligned} |v - P_{V'_{Q_i}}^e(v)|^2 &= |v - P_{V'_{Q_i}}^e(y')|^2 - |P_{V'_{Q_i}}^e(v) - P_{V'_{Q_i}}^e(y')|^2 \\ &\leq (|v - y'| + |y' - P_{V'_{Q_i}}^e(y')|)^2 - |P_{V'_{Q_i}}^e(v) - P_{V'_{Q_i}}^e(y')|^2 \\ &\leq (|v - y'| + C\varepsilon d(R_j))^2 - (1 + \varepsilon)^{-2}|v - y'|^2 \\ &\leq C\varepsilon d(R_j)^2 \end{aligned}$$

and therefore by (F5)

$$|A_i(p) - A_j(p)| \leq |v - P_{V'_{Q_i}}^e(v)| + (1 + \delta)|P(P_{V'_{Q_i}}^e(v)) - P(v)| \leq C\sqrt{\varepsilon}d(R_j).$$

□

For each $i \in I_0$ let $\tilde{\phi}_i : \mathbb{R}^k \rightarrow [0, 1]$ be a C^2 function such that $\tilde{\phi}_i(p) = 1$ for all $p \in 2R_i$, $\tilde{\phi}_i(p) = 0$ for all $p \in \mathbb{R}^k \setminus 3R_i$ and

$$(26) \quad \begin{aligned} |\partial_j \tilde{\phi}_i| &\leq Cd(R_i)^{-1}, \\ |\partial_j \partial_m \tilde{\phi}_i| &\leq Cd(R_i)^{-2} \end{aligned}$$

for all $j, m \in \{1, \dots, k\}$. Set

$$\phi_i(p) = \frac{\tilde{\phi}_i(p)}{\sum_{j \in I_0} \tilde{\phi}_j(p)} \quad \text{for any } p \in U_0 \setminus P(Z), i \in I_0.$$

For each $p \in P(Z)$ there is $x(p)$ such that $P^{-1}(\{p\}) \cap K_0Q(\mathcal{S}) = \{x(p)\}$ by Lemma 4.3. We now define a function $g : U_0 \rightarrow \mathbb{R}^{2n-k}$ by setting

$$g(p) = \begin{cases} \sum_{i \in I_0} \phi_i(p) A_i(p), & \text{if } p \in U_0 \setminus P(Z) \\ P^\perp(x(p)), & \text{if } p \in P(Z). \end{cases}$$

Lemma 5.3. *The function g is $C\sqrt{\delta}$ -Lipschitz.*

Proof. By taking ε/δ small enough we get as in [3, equation (8.19)] that

$$(27) \quad |g(p) - g(q)| \leq 3\sqrt{\delta}|p - q| \quad \text{for } p, q \in 2R_j \cap U_0, j \in I_0.$$

By Lemma 4.3

$$(28) \quad |P^\perp(y) - g(q)| \leq 2\sqrt{\delta(1+\delta)}|P(y) - q| \quad \text{for } q \in P(Z), y \in Q(\mathcal{S}).$$

Let for a while $j \in I_0$, $p \in R_j \cap U_0$, $y \in Q_j$ and $q \in P(Z)$. Then

$$|g(p) - A_j(p)| \leq C\sqrt{\varepsilon}d(R_j)$$

by the definition of g and Lemma 5.2 (because the supports of the functions ϕ_i have bounded overlap by Lemma 5.1), and

$$|A_j(p) - A_j(P(y))| \leq 2\sqrt{\delta}|p - P(y)| \leq C\sqrt{\delta}d(R_j)$$

by (25) and (24). By (F5)

$$|A_j(P(y)) - P^\perp(y)| \leq (2 + \delta)\varepsilon d(Q_j).$$

Here $H(p) = H(p) - H(q) \leq |p - q|$. Thus by (28), (21), (23) and (24) (choosing ε small enough depending on δ and K)

$$(29) \quad |g(p) - g(q)| \leq C\sqrt{\delta}|p - q| \quad \text{for } p \in P(Z), q \in U_0 \setminus P(Z).$$

Lemma now follows easily from (27), (28) and (29). \square

Lemma 5.4. *There is a constant $C = C(K_0)$ such that $P^{-1}(\{p\}) \cap K_0Q(\mathcal{S}) \subset CQ_i$ for all $p \in R_i$, $i \in I_0$.*

Proof. Let $i \in I_0$, $p \in R_i$, $x \in P^{-1}(\{p\}) \cap K_0Q(\mathcal{S})$ and $y \in Q_i$. We may assume that $d(x, y) > d(Q_i)$ (since otherwise $x \in 2Q_i$). Then $d(x, y) > h(y)$ and Lemma 4.3 (by choosing δ small), (24) and (23) yield $d(x, y) \leq 2|p - P(y)| \leq Cd(Q_i)$. \square

Lemma 5.5. *There is a constant $C = C(K_0)$ such that $h(x) \leq CH(P(x))$ for all $x \in K_0Q(\mathcal{S})$.*

Proof. Let $x \in K_0Q(\mathcal{S})$. By Lemma 4.3 one may assume that $P(x) \notin P(Z)$. Let $i \in I_0$ such that $P(x) \in R_i$ (such i exists because P is 1-Lipschitz). Then $h(x) \leq d(x, Q_i) + d(Q_i) \leq Cd(Q_i) \leq CH(P(x))$ by the previous lemma, (23) and (22). \square

Lemma 5.6. *There exists a constant C such that*

$$|P^\perp(x) - g(P(x))| \leq C\sqrt{\varepsilon}h(x)$$

for all $x \in K_0Q(\mathcal{S})$.

Proof. Let $x \in K_0Q(\mathcal{S})$ with $h(x) > 0$. Now $H(P(x)) > 0$ by the previous lemma and so $P(x) \in R_i$ for some $i \in I_0$ and further $x \in CQ_i$ by Lemma 5.4. As in the proof of Lemma 5.3, choosing K large enough depending on K_0 we get $|A_i(P(x)) - P^\perp(x)| \leq (2 + \delta)\varepsilon d(Q_i)$ by (F5). Since also $|g(P(x)) - A_i(P(x))| \leq C\sqrt{\varepsilon}d(R_i)$ by the definition of g and Lemma 5.2, we get the result by (23) and (22). \square

Lemma 5.7. *There is a constant C such that whenever $10R_i \cap 10R_j \neq \emptyset$ for $i, j \in I_0$ then*

$$|\partial_m(A_i(p) - A_j(p))| \leq C\sqrt{\varepsilon} \quad \text{for all } m \in \{1, \dots, k\} \text{ and } p \in \mathbb{R}^k.$$

Proof. Since $\partial_m A_i$ is constant for any m and i it is enough to prove the claim for a fixed $p \in 10R_i \cap 10R_j$. Let $t = d(R_j)$. Then by Lemma 5.2

$$|A_i(p + te_m) - A_j(p + te_m) - (A_i(p) - A_j(p))| \leq C\sqrt{\varepsilon}t,$$

which gives the result, because for any i the quotient $t^{-1}(A_i(p + te_m) - A_i(p))$ does not depend on t . \square

Using (26) and Lemmas 5.1, 5.2 and 5.7 one gets (see [3, Lemma 8.22])

Lemma 5.8. *There is a constant C such that*

$$|\partial_j \partial_m g(p)| \leq \frac{C\sqrt{\varepsilon}}{d(R_i)} \quad \text{for all } j, m \in \{1, \dots, k\} \text{ and } p \in 2R_i \cap \text{int } U_0, i \in I_0.$$

Lemma 5.9. *There is a constant C such that $|g(p)| \leq CK_0\sqrt{\delta}d(Q(\mathcal{S}))$ for all $p \in U_0$.*

Proof. Let $p \in U_0$, $i \in I_0$, $x \in Q_i$ and $y \in V_{Q_i}$ with $d(x, y) \leq \varepsilon d(Q_i)$. By Lemma 2.2

$$|A_i(P(y))| = |P^\perp(y)| \leq |P^\perp(x)| + |P^\perp(y - x)| \leq 3d(x, X_k) + d(x, y) \leq 4\varepsilon d(Q(\mathcal{S})).$$

Further by (25) (and the Lipschitzness of P)

$$\begin{aligned} |A_i(p) - A_i(P(y))| &\leq 2\sqrt{\delta}|p - P(y)| \leq 2\sqrt{\delta}(|p - P(x)| + |P(x) - P(y)|) \\ &\leq 2\sqrt{\delta}((K_0 + 1)d(Q(\mathcal{S})) + \varepsilon d(Q_i)). \end{aligned}$$

Thus $|A_i(p)| \leq CK_0\sqrt{\delta}d(Q(\mathcal{S}))$. The desired estimate for $|g(p)|$ now follows from Lemma 5.1. \square

6. FUNCTION γ FOR \mathcal{S}

In this section we use the same assumptions and notations as in Section 5. Using Lemma 5.9 we extend g from U_0 to a $C\sqrt{\delta}$ -Lipschitz function on \mathbb{R}^k supported in U_{-1} . For $p \in \mathbb{R}^k$ and $t > 0$ set

$$\gamma(p, t) = t^{-k-1} \inf_a \int_{B^k(p, t)} |g(u) - a(u)| du,$$

where the infimum is taken over all affine functions $a : \mathbb{R}^k \rightarrow \mathbb{R}^{2n-k}$. Choosing δ small one has by Lemma 5.3

$$(30) \quad \gamma(p, t) \leq 2t^{-k-1} \inf_M \int_{B^k(p, t)} d_e((u, g(u)), M) du,$$

where the infimum is taken over all k -planes $M \subset \mathbb{R}^{2n}$. We follow [3, Section 13] and proof the next lemma.

Lemma 6.1. *Let $T = K_0 d(Q(\mathcal{S}))/2$. There exists a constant $C = C(K_0)$ such that*

$$\int_0^T \int_{U_1} \gamma(p, t)^2 dp \frac{dt}{t} \leq C\varepsilon\mu(Q(\mathcal{S})) + C\varepsilon^{-6k} \int_{K_0 Q(\mathcal{S})} \int_{h(x)/K_0}^T \beta_1(x, K_0 t)^2 \frac{dt}{t} d\mu x.$$

Notice that $2R_i \subset U_0$ for all $i \in I_1$ by (20) and (21). Using Lemma 5.8 and Taylor's theorem one gets (see [3, Lemma 13.7])

Lemma 6.2. *There is a constant C such that*

$$\sum_{i \in I_1} \int_0^{d(R_i)} \int_{R_i} \gamma(p, t)^2 dp \frac{dt}{t} \leq C\varepsilon\mu(Q(\mathcal{S})).$$

We now assume that $p \in U_1$ and $H(p)/60 < t \leq T$. Choose $z(p, t) \in Q(\mathcal{S})$ such that $|p - P(z(p, t))| \leq 60t$ (see (20)) and let $z \in B(z(p, t), t) \cap E$. Further let $V_{p, t} \in \mathcal{V}^k$ be such that

$$(31) \quad \int_{B(z, K_0 t)} d(x, V_{p, t}) d\mu x \leq 2(K_0 t)^{k+1} \beta_1(z, K_0 t).$$

By (30)

$$(32) \quad 2^{-1} t^{k+1} \gamma(p, t) \leq \int_{B^k(p, t)} d_e((u, g(u)), V'_{p, t}) du.$$

If $u \in B^k(p, t) \cap P(Z)$ then $P^{-1}(\{u\}) \cap Q(\mathcal{S}) = \{x'\}$, where $x' = (u, g(u))$ and $h(x) = 0$. Since $z \in K_0 Q(\mathcal{S})$ (by choosing $K_0 \geq 2$),

$$d(x, z) \leq (1 + 2\delta)|u - P(z)| \leq (1 + 2\delta)(|u - p| + |p - P(z(p, t))| + t) \leq Ct$$

by Lemma 4.3. Thus

$$(33) \quad \int_{B^k(p,t) \cap P(Z)} d_e((u, g(u)), V'_{p,t}) du \leq \int_{B(z,Ct)} d_e(x', V'_{p,t}) d\mu x.$$

Let $I(p, t) = \{i \in I_0 : R_i \cap B^k(p, t) \neq \emptyset\}$.

Lemma 6.3. *There is a constant $C = C(K_0)$ such that for any $i \in I(p, t)$*

$$d_e((u, g(u)), V'_{p,t}) \leq d_e((u, g(u)), V'_{Q_i}) + \sup\{d_e(w', V'_{p,t}) : w \in V_{Q_i} \cap Q_i(Cd(Q_i))\}$$

for all $u \in R_i \cap U_0$.

Proof. Let $i \in I(p, t)$ and $u \in R_i$. Denote $y = (u, g(u))$. Let $w \in V_{Q_i}$ be such that $|y - w'| = d_e(y, V'_{Q_i})$. Further let $v \in V_{p,t}$ be such that $|w' - v'| = d_e(w', V'_{p,t})$. Then

$$d_e(y, V'_{p,t}) \leq |y - v'| \leq |y - w'| + |w' - v'| \leq d_e(y, V'_{Q_i}) + d_e(w', V'_{p,t}).$$

By the definition of g and Lemma 5.2

$$|u - P(w)| \leq |y - w'| \leq |y - (u, A_i(u))| \leq C\sqrt{\varepsilon}d(R_i).$$

Choosing $q \in Q_i$ and $v_q \in V_{Q_i}$ with $d(v_q, q) \leq \varepsilon d(Q_i)$ one has by (F5), (24) and (23)

$$\begin{aligned} d(w, q) &\leq d(w, v_q) + d(v_q, q) \leq (1 + \delta)|P(w) - P(v_q)| + d(v_q, q) \\ &\leq (1 + \delta)(|P(w) - u| + |u - P(q)| + |P(q) - P(v_q)|) + d(v_q, q) \leq Cd(Q_i). \end{aligned}$$

□

For any $i \in I(p, t)$

$$(34) \quad \int_{B^k(p,t) \cap R_i} d_e((u, g(u)), V'_{Q_i}) du \leq \int_{B^k(p,t) \cap R_i} |g(u) - A_i(u)| du \leq C\sqrt{\varepsilon}d(R_i)^{k+1}$$

by Lemma 5.2 and the definitions of A_i and g . Recall that $B^k(p, t) \subset U_0$ (because $p \in U_1$ and $t \leq T$).

Lemma 6.4. *For any constant C' there is a constant C such that for any $i \in I(p, t)$*

$$d_e(w', V'_{p,t}) \leq C\varepsilon d(R_i) + C\varepsilon^{-3k} \left(\int_{2Q_i} d_e(x', V'_{p,t})^{1/3} d\mu x \right)^3$$

for all $w \in V_{Q_i} \cap Q_i(C'd(Q_i))$.

Proof. Let $i \in I(p, t)$ and $w \in V_{Q_i} \cap Q_i(C'd(Q_i))$. If $y_0, \dots, y_k \in V_{Q_i} \cap Q_i(\varepsilon d(Q_i))$ are as in Lemma 3.1 with $Q = Q_i$, then obviously (by (6))

$$d_e(w', V'_{p,t}) \leq Cd_e(y'_{j_0}, V'_{p,t}),$$

where $d_e(y'_{j_0}, V'_{p,t}) = \max\{d_e(y'_j, V'_{p,t}) : j \in \{0, \dots, k\}\}$. Let $z_0 \in Q_i \cap B(y_{j_0}, 2\varepsilon d(Q_i))$. Then

$$d_e(y'_{j_0}, V'_{p,t}) \leq d_e(x', V'_{p,t}) + 3\varepsilon d(Q_i)$$

for all $x \in B := B(z_0, \varepsilon d(Q_i))$, and we have

$$\begin{aligned} \mu(B)d_e(w', V'_{p,t})^{1/3} &\leq C \int_B (d_e(x', V'_{p,t}) + 3\varepsilon d(Q_i))^{1/3} d\mu x \\ &\leq C\mu(B)(3\varepsilon d(Q_i))^{1/3} + C \int_B d_e(x', V'_{p,t})^{1/3} d\mu x. \end{aligned}$$

The claim now follows from the regularity, (14) and (23). □

Lemma 6.5. *There is a constant C such that*

$$\sum_{i \in I(p,t)} \mathcal{L}^k(R_i) \left(\int_{2Q_i} d_e(x', V'_{p,t})^{1/3} d\mu x \right)^3 \leq C \int_{B(z,Ct)} d_e(x', V'_{p,t}) d\mu x.$$

(Here \mathcal{L}^k is the Lebesgue measure on \mathbb{R}^k .)

Proof. For any $i \in I$ define $N_i : \mathbb{H}^n \rightarrow \mathbb{R}$ by setting

$$N_i = \sum_{j \in J(i)} \chi_{2Q_j}$$

where $J(i) = \{j \in I : d(R_j) \leq d(R_i) \text{ and } 2Q_i \cap 2Q_j \neq \emptyset\}$. Notice that $N_i(x) \geq 1$ for all $x \in 2Q_i$, $i \in I$. Let $l, m \in I$. If $2Q_l \cap 2Q_m \neq \emptyset$ and $d(R_l) \leq d(R_m)$, then $d(R_l \cup R_m) \leq d(P(2Q_l) \cup R_l) + d(P(2Q_m) \cup R_m) \leq Cd(R_m)$ by (23) and (24). Hence by (14)

$$\int_{2Q_i} N_i(x) d\mu x \leq \sum_{j \in J(i)} \mu(2Q_j) \leq C \sum_{j \in J(i)} \mathcal{L}^k(R_j) \leq C\mathcal{L}^k(R_i),$$

and further by Hölder's inequality

$$(35) \quad \left(\int_{2Q_i} d_e(x', V'_{p,t})^{1/3} d\mu x \right)^3 \leq C\mathcal{L}^k(R_i)^2 \int_{2Q_i} d_e(x', V'_{p,t}) N_i(x)^{-2} d\mu x$$

for any $i \in I$. If $x \in 2Q_l \cap 2Q_m$ and $N_l(x) = N_m(x)$ for $l, m \in I$, then by the definition necessarily $d(R_l) = d(R_m)$ and further (see above)

$$(36) \quad \sum_{i \in I} \chi_{2Q_i}(x) N_i(x)^{-2} = \sum_{m=1}^{\infty} \left(m^{-2} \sum_{i \in J(x, m)} \chi_{2Q_i}(x) \right) \leq C,$$

where $J(x, m) = \{i \in I : N_i(x) = m\}$. By (35), (36) and (14)

$$(37) \quad \begin{aligned} & \sum_{i \in I(p, t)} \mathcal{L}^k(R_i) \left(\int_{2Q_i} d_e(x', V'_{p,t})^{1/3} d\mu x \right)^3 \\ & \leq C \sum_{i \in I(p, t)} \int_{2Q_i} d_e(x', V'_{p,t}) N_i(x)^{-2} d\mu x \leq C \int_{\bigcup_{i \in I(p, t)} 2Q_i} d_e(x', V'_{p,t}) d\mu x. \end{aligned}$$

Let $i \in I(p, t)$ and $x \in 2Q_i$. Since $H(u) \leq H(p) + t \leq 61t$ for all $u \in B(p, t)$, one has $d(R_i) \leq 4t$ by (21). Thus by (23) and (24)

$$|P(x) - P(z)| \leq d(P(2Q_i) \cup R_i) + d(R_i \cup \{p\}) + |p - P(z(p, t))| + t \leq Ct.$$

Since $h(x) \leq 2d(Q_i) \leq Ct$ (by the definition of h and (23)), Lemma 4.3 implies $d(x, z) \leq Ct$. (Namely, if $d(x, z) \geq Ct$ then $d(x, z) \leq (1 + 2\delta)|P(x) - P(z)|$ by Lemma 4.3.) The claim now follows from (37). \square

Now (32), (33), Lemma 6.3, (34), Lemma 6.4, Lemma 6.5 and (31) give (by choosing $\varepsilon < 1$ and K_0 large enough)

$$\gamma(p, t) \leq C\varepsilon^{-3k} \beta_1(z, K_0 t) + C\sqrt{\varepsilon} t^{-k-1} \sum_{i \in I(p, t)} d(R_i)^{k+1}$$

and further by the regularity

$$(38) \quad \gamma(p, t)^2 \leq C\varepsilon^{-6k} t^{-k} \int_{B(z(p, t), t)} \beta_1(z, K_0 t)^2 d\mu z + C\varepsilon t^{-2(k+1)} \left(\sum_{i \in I(p, t)} d(R_i)^{k+1} \right)^2$$

for some constant $C = C(K_0)$.

If $p \in U_1$, $H(p)/60 < t \leq T$ and $z \in B(z(p, t), t) \cap E$ then $z \in K_0 Q(\mathcal{S})$ and $|p - P(z)| \leq |p - P(z(p, t))| + |P(z(p, t)) - P(z)| \leq Ct$. We also have $h(z) \leq Ct$. (Namely, choose $\tilde{u} \in P^{-1}(\{p\})$ with $h(\tilde{u}) \leq H(p) + t$. Then let $u \in Q(\mathcal{S})$ be with $d(u, \tilde{u}) \leq h(\tilde{u}) + t$. If $d(z, u) > h(u)$

then $d(z, u) \leq 2|P(z) - P(u)| \leq 2(|P(z) - p| + d(u, \tilde{u})) \leq Ct$ by Lemma 4.3. In any case $h(z) \leq h(u) + d(z, u) \leq Ct$.) Thus

$$\begin{aligned}
(39) \quad & \int_{U_1} \int_{H(p)/60}^T t^{-k} \int_{B(z(p,t),t)} \beta_1(z, K_0 t)^2 d\mu z \frac{dt}{t} dp \\
& \leq \int_{K_0 Q(\mathcal{S})} \int_{h(z)/C}^T t^{-k} \left(\int_{B^k(P(z), Ct)} dp \right) \beta_1(z, K_0 t)^2 \frac{dt}{t} d\mu z \\
& \leq C \int_{K_0 Q(\mathcal{S})} \int_{h(z)/C}^T \beta_1(z, K_0 t)^2 \frac{dt}{t} d\mu z
\end{aligned}$$

Since $d(p, R_i) \leq t$ and $d(R_i) \leq Ct$ if $p \in U_1$, $H(p)/60 < t \leq T$ and $i \in I(p, t)$ (as mentioned after (37))

$$\begin{aligned}
(40) \quad & C^{-1} \int_{U_1} \int_{H(p)/60}^T t^{-2(k+1)} \left(\sum_{i \in I(p,t)} d(R_i)^{k+1} \right)^2 \frac{dt}{t} dp \\
& \leq \int_{U_1} \int_{H(p)/60}^T \sum_{i \in I(p,t)} d(R_i)^{k+1} \frac{dt}{t^{k+2}} dp \leq \sum_{i \in I_0} d(R_i)^{k+1} \int_{d(R_i)/C}^T \int_{R_i(t)} dp \frac{dt}{t^{k+2}} \\
& \leq \sum_{i \in I_0} d(R_i)^{k+1} \int_{d(R_i)/C}^\infty (d(R_i) + 2t)^k \frac{dt}{t^{k+2}} \leq C \sum_{i \in I_0} d(R_i)^k \leq C\mu(Q(\mathcal{S})),
\end{aligned}$$

where the last constant C depends on K_0 . The last inequality follows from (14) because $d(R_i) \leq K_0 d(Q(\mathcal{S}))/20$ by (20) and (21) and therefore $R_i \subset B^k(P(x_0), 2K_0 d(Q(\mathcal{S})))$ for any $i \in I_0$. From (38), (39) and (40) one now gets

$$\int_{U_1} \int_{H(p)/60}^T \gamma(p, t)^2 \frac{dt}{t} dp \leq C\varepsilon\mu(Q(\mathcal{S})) + C\varepsilon^{-6k} \int_{K_0 Q(\mathcal{S})} \int_{h(z)/C}^T \beta_1(z, K_0 t)^2 \frac{dt}{t} d\mu z.$$

Combining this with Lemma 6.2 we get Lemma 6.1 because $U_1 \subset P(Z) \cup \bigcup_{i \in I_1} R_i$, $H(P(Z)) = \{0\}$ and $60t > H(p)$ by (22) whenever $p \in R_i$ with $d(R_i) < t$ and $i \in I_0$.

7. ESTIMATE FOR $\mathcal{S} \in \mathfrak{F}_3$

In this section we use the same assumptions and notations as in Section 4. We follow [3, Sections 14 and 11] or [7, Section 5] and prove the next lemma.

Lemma 7.1. *Let $\mathcal{S} \in \mathfrak{F}_3$. Then*

$$\int_{K_0 Q(\mathcal{S})} \int_{h(x)/K_0}^{K_0 d(Q(\mathcal{S}))} \beta_1(x, K_0 t)^2 \frac{dt}{t} d\mu x > \varepsilon^{6k+1} \mu(Q(\mathcal{S})).$$

Fix $\mathcal{S} \in \mathfrak{F}_3$ and suppose to the contrary that the claim is not true for \mathcal{S} . Since the translations and the rotations are isometries we can assume that $V_{Q(\mathcal{S})} = X_k$ (see 2.1). We use the same notations as in Section 5. By Lemma 6.1

$$(41) \quad \int_0^{K_0 d(Q(\mathcal{S}))/2} \int_{U_1} \gamma(p, t)^2 dp \frac{dt}{t} \leq C(K_0)\varepsilon\mu(Q(\mathcal{S})).$$

Let $\nu : \mathbb{R}^k \rightarrow \mathbb{R}$ be a radial C^∞ function supported in $B^k(0, 1)$ such that $\int_{\mathbb{R}^k} f\nu = 0$ for any affine function $f : \mathbb{R}^k \rightarrow \mathbb{R}$ and

$$\int_0^\infty |\hat{\nu}(tp)|^2 \frac{dt}{t} = 1$$

for all $p \in \mathbb{R}^k \setminus \{0\}$. Denote $\nu_t(p) = t^{-k}\nu(t^{-1}p)$ for any $p \in \mathbb{R}^k$ and $t > 0$. Using Calderón's formula one can write

$$g(p) = \int_0^\infty (\nu_t * \nu_t * g)(p) \frac{dt}{t}$$

for any $p \in \mathbb{R}^k$. (Notice that the above integral exists and depends continuously on p , because g is Lipschitz and has compact support.) Set $L = K_0 d(Q(\mathcal{S}))/5$ and write $g = g_1 + g_2$, where $g_1, g_2 : \mathbb{R}^k \rightarrow \mathbb{R}^{2n-k}$ are defined by

$$\begin{aligned} g_1(p) &= \int_L^\infty (\nu_t * \nu_t * g)(p) \frac{dt}{t} + \int_0^L (\nu_t * (\chi_{\mathbb{R}^k \setminus U_1} \cdot (\nu_t * g)))(p) \frac{dt}{t}, \\ g_2(p) &= \int_0^L (\nu_t * (\chi_{U_1} \cdot (\nu_t * g)))(p) \frac{dt}{t}. \end{aligned}$$

Lemma 7.2. *There is a constant C such that $|\partial_j g_1(p)| \leq C\sqrt{\delta}$ and $|\partial_i \partial_j g_1(p)| \leq C\sqrt{\delta}/L$ for any $i, j \in \{1, \dots, k\}$ and $p \in U_2$.*

Proof. We first notice that

$$(42) \quad \int_0^L (\nu_t * (\chi_{\mathbb{R}^k \setminus U_1} \cdot (\nu_t * g)))(p) \frac{dt}{t} = 0$$

for all $p \in U_{9/5} \supset U_2$. Set

$$\varphi(q) = \int_L^\infty (\nu_t * \nu_t)(q) \frac{dt}{t}$$

for all $q \in \mathbb{R}^k$. By (42) one has $g_1(p) = (\varphi * g)(p)$ for all $p \in U_{9/5}$. Fix $i, j \in \{1, \dots, k\}$ and $p \in U_2$. Since $|\nu_t * \nu_t| \leq Ct^{-k}$ for any $t > 0$, one has $|\varphi| \leq CL^{-k}$. Further $|\nabla \varphi| \leq CL^{-k-1}$. (Here C depends on ν .) Particularly

$$(43) \quad \int_{U_{-1}} |\varphi(p - q)| dq \leq C \quad \text{and} \quad \int_{U_{-1}} |\partial_i \varphi(p - q)| dq \leq \frac{C}{L}.$$

Since φ is bounded and g has compact support, Lemma 5.3 and the dominated convergence give $\partial_j g_1(p) = (\varphi * \partial_j g)(p)$. (Notice that g is differentiable almost everywhere by Rademacher's theorem.) Thus, since $\partial_i \varphi$ is bounded and $\partial_j g$ is compactly supported and bounded, we further have $\partial_i \partial_j g_1(p) = (\partial_i \varphi * \partial_j g)(p)$. Since $\partial_j g$ is supported in U_{-1} , the claim now follows from Lemma 5.3 and (43). \square

Define $g_{2,m}$ for any $m \in \mathbb{N}$ by setting

$$g_{2,m}(p) = \int_{1/m}^L (\nu_t * (\chi_{U_1} \cdot (\nu_t * g)))(p) \frac{dt}{t}$$

for all $p \in \mathbb{R}^k$. Then $g_{2,m} \rightarrow g_2$ uniformly as $m \rightarrow \infty$. We also have that $g_{2,m} \rightarrow g_2$ in L^2 because g_2 is bounded and $\text{spt } g_{2,m} \subset U_0$ for all m . Now

$$\partial_j g_{2,m}(p) = \int_{1/m}^L (\partial_j \nu_t * (\chi_{U_1} \cdot (\nu_t * g)))(p) \frac{dt}{t}$$

for any $p \in \mathbb{R}^k$, $j \in \{1, \dots, k\}$ and $m \in \mathbb{N}$. Using this we find a constant $C = C(\nu)$ such that

$$(44) \quad \int_{\mathbb{R}^k} |\partial_j g_{2,m}(p)|^2 dp \leq C \int_{1/m}^L \int_{U_1} \gamma(p, t)^2 dp \frac{dt}{t}$$

for any $j \in \{1, \dots, k\}$ and $m \in \mathbb{N}$ (see [3] or one dimensional case [7, page 863]). Particularly $(g_{2,m})_m$ is a bounded sequence in the Sobolev space $W^{1,2}$ (by (41)) and a subsequence of $(\partial_j g_{2,m})_m$ converges weakly in L^2 to $\partial_j g_2$. Thus by (44) and (41)

$$(45) \quad \int_{\mathbb{R}^k} |\partial_j g_2(p)|^2 dp \leq C(K_0) \varepsilon \mu(Q(\mathcal{S}))$$

for any $j \in \{1, \dots, k\}$.

Define a function $N : \mathbb{R}^k \rightarrow \mathbb{R}$ by setting

$$N(p) = \sup_B \frac{m_B(|g_2 - m_B(g_2)|)}{d(B)},$$

where the supremum is taken over all balls B in \mathbb{R}^k containing p and having (positive) radius at most L . Here we use the notation $m_B(f) = \int_B f$ for locally integrable functions $f : \mathbb{R}^k \rightarrow \mathbb{R}$. From Poincaré's inequality and the Hardy-Littlewood maximal inequality one now gets (see [3, page 75] or one dimensional case [7, Lemma 5.3])

$$(46) \quad \int_{\mathbb{R}^k} N(p)^2 dp \leq C \max_j \int_{\mathbb{R}^k} |\partial_j g_2(p)|^2 dp.$$

Since $g_2|_{U_2}$ is $C\sqrt{\delta}$ -Lipschitz by Lemmas 5.3 and 7.2, one gets (see [3, Lemma 11.8]) that for any closed ball $B \subset U_2$

$$(47) \quad \sup_{p \in B} |g_2(p) - m_B(g_2)| \leq C \delta^{\frac{k}{2(k+1)}} d(B) N(q)^{\frac{1}{k+1}}$$

whenever $q \in B$. Set $F = \{q \in U_3 : N(q)^3 \leq \varepsilon\}$. Using (47), Lemma 7.2 and Taylor's theorem one gets (see [3, Lemma 11.9]) that

$$(48) \quad \sup_{p \in B^k(p_0, r)} |g(p) - g(p_0) - Dg_1(p_0)(p - p_0)| \leq C \delta^{\frac{k}{2(k+1)}} \varepsilon^{\frac{1}{3(k+1)}} r + C\sqrt{\delta} L^{-1} r^2$$

whenever $r \leq L/4$ and $B^k(p_0, r) \cap F \neq \emptyset$. For any $p \in U_2$ let $\Delta_p \subset \mathbb{R}^{2n}$ be the k -plane which is the graph of the affine function $q \mapsto g(p) + Dg_1(p)(q - p)$.

Lemma 7.3. *If $Q \in m_2(\mathcal{S})$ then $d(P(Q), F) > d(Q)$.*

Proof. Suppose to the contrary that $Q \in m_2(\mathcal{S})$ with $d(P(Q), F) \leq d(Q)$. Since $\mathcal{C}(Q) \subset \mathcal{G}$ by definition of $m_2(\mathcal{S})$, we get by (F7) a contradiction $Q \notin \min(\mathcal{S})$ by showing that $\angle(V_R, V_{Q(\mathcal{S})}) \leq 1 + \delta$ for all $R \in \mathcal{C}(Q)$. For that reason, fix $R \in \mathcal{C}(Q)$. If now $2Kd(R) \geq d(Q(\mathcal{S}))$ then $\angle(V_R, V_{Q(\mathcal{S})}) \leq 1 + \varepsilon$ by Lemma 3.2. Thus we may assume that $2Kd(R) < D^2\alpha d(Q(\mathcal{S}))$. Pick $x \in Q$ and set $r = 3d(Q)$. Now $r < L/K \leq L/4$ (by (14) choosing K_0 and K large enough) and $B^k(P(x), r) \cap F \neq \emptyset$. By this, Lemma 5.6 and (48)

$$(49) \quad \begin{aligned} d_e(y', \Delta_{P(x)}) &\leq |P^\perp(y) - g(P(x)) - Dg_1(P(x))(P(y) - P(x))| \\ &\leq C\sqrt{\varepsilon}h(y) + C\delta^{\frac{k}{2(k+1)}} \varepsilon^{\frac{1}{3(k+1)}} r + C\sqrt{\delta}K^{-1}r \end{aligned}$$

for any $y \in 2Q$. Let $y_0, \dots, y_k \in V_R \cap Q(\varepsilon^2 d(R))$ be as in Lemma 3.1 (recalling that $R \in \mathcal{G}$). Then by (6), (14) and (49)

$$d_e(y'_j, \Delta_{P(x)}) \leq C \left(\sqrt{\varepsilon} + \delta^{\frac{k}{2(k+1)}} \varepsilon^{\frac{1}{3(k+1)}} + \sqrt{\delta}K^{-1} \right) d(R)$$

for any $j \in \{1, \dots, k\}$. Further $d_e(y'_{i+1}, L'_i) > cd(R)$ for all $i \in \{0, \dots, k-1\}$, where L_i and c are as in Lemma 3.1. Thus the euclidean angle between V'_R and $\Delta_{P(x)}$ is less than $\delta/9$ by taking ε small enough and K large enough depending on δ . Let Q^* be the minimal cube in \mathcal{S} such that $Q \subset Q^*$ and $2Kd(Q^*) \geq d(Q(\mathcal{S}))$. Then $2Kd(Q^*) < D^2\alpha d(Q(\mathcal{S}))$ (by (14)) and by the above argument the angle between V'_{Q^*} and $\Delta_{P(x)}$ is also less than $\delta/9$. Now $\angle(V_{Q^*}, V_{Q(\mathcal{S})}) \leq 1 + \varepsilon$ by Lemma 3.2. Choosing ε/δ small the euclidean angle between V'_{Q^*} and $V'_{Q(\mathcal{S})}$ is less than $\delta/9$ (by Lemma 2.3). Thus the angle between V'_R and $V'_{Q(\mathcal{S})}$ is less than $\delta/3$ and so $\angle(V_R, V_{Q(\mathcal{S})}) \leq 1 + \delta$ (by choosing δ small). \square

For each $Q \in m_2(\mathcal{S})$ pick $x_Q \in Q$. By the $5r$ -covering lemma we find $\mathcal{T} \subset m_2(\mathcal{S})$ such that the balls $B(x_Q, 3d(Q))$, $Q \in \mathcal{T}$, are disjoint and

$$G := \bigcup_{Q \in m_2(\mathcal{S})} Q \subset \bigcup_{Q \in \mathcal{T}} B(x_Q, 15d(Q)).$$

Since $d(x_Q, x_R) > 3\max\{d(Q), d(R)\} \geq h(x_Q)$ for any distinct $Q, R \in \mathcal{T}$, Lemma 4.3 gives (by choosing δ small) that the balls $B^k(P(x_Q), d(Q))$, $Q \in \mathcal{T}$, are also disjoint. Further

$B^k(P(x_Q), d(Q)) \subset U_3 \setminus F$ for any $Q \in m_2(\mathcal{S})$ by Lemma 7.3. Hence by (46) and (45)

$$\begin{aligned} \mu(G) &\leq \sum_{Q \in \mathcal{T}} \mu(B(x_Q, 15d(Q))) \leq 15^k C_E \sum_{Q \in \mathcal{T}} d(Q)^k \\ &\leq C \mathcal{L}^k \left(\bigcup_{Q \in \mathcal{T}} B^k(P(x_Q), d(Q)) \right) \leq C \mathcal{L}^k(U_3 \setminus F) \\ &\leq C \varepsilon^{-2/3} \int_{U_3 \setminus F} N(p)^2 dp \leq C(K_0) \varepsilon^{1/3} \mu(Q(\mathcal{S})). \end{aligned}$$

Choosing ε small enough this means that $\mathcal{S} \notin \mathfrak{F}_3$ which is a contradiction.

8. END OF THE PROOF

In this section we follow [3, Sections 12 and 16] and use the same assumptions and notations as in Section 3 and assume further that (3) is satisfied. Now the assumptions of Sections 4 and 7 are also satisfied (see Section 3). From this on the constants C may depend without special mention on ε , K , δ or K_0 .

Lemma 8.1. *There is a constant C such that*

$$\sum_{\substack{\mathcal{S} \in \mathfrak{F} \\ Q(\mathcal{S}) \subset R}} \mu(Q(\mathcal{S})) \leq C \mu(R) \quad \text{for all } R \in \Delta.$$

Proof. For any $\mathcal{S} \in \mathfrak{F}$ denote

$$E_{\mathcal{S}} = \{ (x, t) \in K_0 Q(\mathcal{S}) \times]0, K_0 d(Q(\mathcal{S}))[: h_{\mathcal{S}}(x) < K_0 t \}.$$

Suppose for a while that $\mathcal{S} \in \mathfrak{F}$ and $(x, t) \in E_{\mathcal{S}}$. Then $d(x, Q) + d(Q) < K_0 t < K_0^2 d(Q(\mathcal{S}))$ for some $Q \in \mathcal{S}$. Let Q^* be the minimal cube in \mathcal{S} such that $Q \subset Q^*$ and $K_0 d(Q^*) > t$. Then (by (14)) $K_0 d(Q^*) \leq D^2 \alpha t$ and $C \mu(Q^*) > t^k$. Since trivially $d(x, Q^*) \leq d(x, Q) < K_0 t$ we conclude by the regularity that there is a constant C such that

$$(50) \quad \sum_{\mathcal{S} \in \mathfrak{F}} \chi_{E_{\mathcal{S}}}(x, t) \leq C \quad \text{for all } (x, t) \in E \times \mathbb{R}.$$

Using Lemma 7.1, (50), (3) and (14) one gets for any $R \in \Delta$

$$\begin{aligned} \sum_{\substack{\mathcal{S} \in \mathfrak{F}_3 \\ Q(\mathcal{S}) \subset R}} \mu(Q(\mathcal{S})) &< \varepsilon^{-6k-1} \sum_{\substack{\mathcal{S} \in \mathfrak{F}_3 \\ Q(\mathcal{S}) \subset R}} \int_{K_0 Q(\mathcal{S})} \int_{h_{\mathcal{S}}(x)/K_0}^{K_0 d(Q(\mathcal{S}))} \beta_1(x, K_0 t)^2 \frac{dt}{t} d\mu x \\ &\leq C \int_0^{K_0 d(R)} \int_{K_0 R} \beta_1(x, K_0 t)^2 d\mu x \frac{dt}{t} \leq C \mu(R). \end{aligned}$$

The claim now follows from Lemma 4.2. □

Theorem 1.1 follows from the following lemma.

Lemma 8.2. *For any $\eta > 0$ there is $C > 0$ such that for all $z \in E$ and $r > 0$ there is $F \subset \mathbb{H}^n$ and a C -bilipschitz mapping $f : F \rightarrow \mathbb{R}^k$ such that $\mu(B(z, r) \setminus F) \leq \eta r^k$.*

Proof. Let $\eta > 0$, $z \in E$ and $r \in \mathbb{R}$ with $0 < r \leq d(E)$. Let $m_0 \in \mathbb{Z}$ be such that $D\alpha^{m_0-1} < r \leq D\alpha^{m_0}$. Set

$$\begin{aligned} \mathcal{R}_0 &= \{ Q \in \Delta_{m_0} : Q \cap B(z, r) \neq \emptyset \}, \\ \tilde{\mathfrak{F}} &= \{ \mathcal{S} \cap \{ Q : Q \subset R \} : \mathcal{S} \in \mathfrak{F}, R \in \mathcal{R}_0 \} \setminus \{ \emptyset \}. \end{aligned}$$

Further let

$$\tilde{\Delta} = \bigcup_{j=-\infty}^{m_0} \tilde{\Delta}_j \quad \text{and} \quad \tilde{\mathcal{G}} = \mathcal{G} \cap \tilde{\Delta},$$

where

$$\tilde{\Delta}_j = \left\{ Q \in \Delta_j : Q \subset \bigcup_{R \in \mathcal{R}_0} R \right\}.$$

One easily sees that (F1), (F2) and Lemma 8.1 remain valid if Δ_j , Δ , \mathcal{G} and \mathfrak{F} are replaced by $\tilde{\Delta}_j$, $\tilde{\Delta}$, $\tilde{\mathcal{G}}$ and $\tilde{\mathfrak{F}}$. (For Lemma 8.1 this is because the new maximal cubes $Q(\mathcal{S})$, $\mathcal{S} \in \tilde{\mathfrak{F}} \setminus \mathfrak{F}$, belong to \mathcal{R}_0 .)

For any $Q \in \tilde{\Delta}$ denote

$$\sigma(Q) = \{ x \in Q : d(x, E \setminus Q) \leq \tau \alpha^{j_Q} \},$$

where τ is a small positive constant fixed later. Let $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, where $\mathcal{T}_1 = \tilde{\Delta} \setminus \mathcal{G}$, $\mathcal{T}_2 = \{ Q(\mathcal{S}) : \mathcal{S} \in \tilde{\mathfrak{F}} \}$ and $\mathcal{T}_3 = \bigcup_{\mathcal{S} \in \tilde{\mathfrak{F}}} \min(\mathcal{S})$. For any $Q \in \mathcal{T}$ set $\ell(Q) = \#\{ R \in \mathcal{T} : R \neq Q \subset R \}$. Define

$$F = \left(B(z, r) \cap \bigcap_{j \in \mathbb{Z}} \bigcup_{Q \in \Delta_j} Q \right) \setminus (F_1 \cup F_2),$$

where

$$F_1 = \bigcup_{Q \in \mathcal{T}} \sigma(Q) \quad \text{and} \quad F_2 = \bigcup_{\substack{Q \in \mathcal{T} \\ \ell(Q) > M}} Q.$$

Using (8), (F1), Lemma 8.1 and (13) and choosing τ small enough and M large enough depending on η one gets $\mu(B(z, r) \setminus F) \leq \mu(F_1 \cup F_2) \leq \eta r^k$ (see [3, pages 102–103]).

For the definition on f we first define a map $t : \mathcal{T} \rightarrow \mathcal{P}(\mathbb{R}^k)$ recursively as follows. First, for each $Q \in \mathcal{R}_0 = \tilde{\Delta}_{m_0}$ let $t(Q)$ be a cube in \mathbb{R}^k with side length α^{j_Q} . Since $\#\mathcal{R}_0 \leq 2^k C_E D^{k+1}$ by the regularity and (14), one can choose the cubes $t(Q)$ so that

$$(51) \quad \alpha^{m_0} \leq d(t(Q_1), t(Q_2)) \leq C \alpha^{m_0}$$

for any distinct $Q_1, Q_2 \in \mathcal{R}_0$ where C is a constant (depending only on C_E , D and k).

Let $Q \in \mathcal{T}$. Assume by recursion that a cube $t(Q) \subset \mathbb{R}^k$ has already been defined such that

$$(52) \quad l(t(Q)) = c_1^{\ell(Q)} \alpha^{j_Q},$$

where $l(G) = d(G)/\sqrt{k}$ for $G \subset \mathbb{R}^k$ and $c_1 > 0$ is a small constant to be chosen later. Assume first that $Q \in \mathcal{T}_1 \cup \mathcal{T}_3$. Then $\ell(R) = \ell(Q) + 1$ for all $R \in \mathcal{C}(Q) \subset \mathcal{T}_1 \cup \mathcal{T}_2$. Since further $j_R < j_Q$ for all $R \in \mathcal{C}(Q)$ and $\#\mathcal{C}(Q) \leq D^2 \alpha^k$ (by (14)), one can choose by (52) the cubes $t(R)$, $R \in \mathcal{C}(Q)$, such that

$$(53) \quad l(t(R)) = c_1^{\ell(R)} \alpha^{j_R},$$

$$(54) \quad t(R) \subset t(Q),$$

$$(55) \quad d(t(R), t(R_1)) \geq c_1^{\ell(Q)+1} \alpha^{j_Q}$$

for all distinct $R, R_1 \in \mathcal{C}(Q)$ provided c_1 is small enough (depending on D , α and k).

Assume now that $Q \in \mathcal{T}_2$ i.e. $Q = Q(\mathcal{S})$ for some $\mathcal{S} \in \tilde{\mathfrak{F}}$. Denote $W_Q = V_{Q(\mathcal{S}_0)}$ where $\mathcal{S}_0 \in \mathfrak{F}$ is such that $\mathcal{S} = \mathcal{S}_0 \cap \{Q : Q \subset Q_0\}$ for some $Q_0 \in \mathcal{R}_0$. By the 1-Lipschitzness of P_{W_Q} , (11), (6) and (52) there is a function $\phi_Q : W_Q \rightarrow \mathbb{R}^k$ such that

$$(56) \quad \phi_Q(P_{W_Q}(Q)) \subset 2^{-1} t(Q),$$

$$(57) \quad |\phi_Q(p) - \phi_Q(q)| = \frac{1}{4D} c_1^{\ell(Q)} d(p, q)$$

for all $p, q \in W_Q$. Here $\lambda G = \{x \in G : d(x, \mathbb{R}^k \setminus G) \geq (1 - \lambda)l(G)/2\}$ for $\lambda \in \mathbb{R}$ and a cube $G \subset \mathbb{R}^k$. For any $R \in \min(\mathcal{S}) \setminus \{Q\}$ let $t(R)$ be a cube satisfying (53) and centered at $\phi_Q(P_{W_Q}(x_R))$, where $x_R \in R$ is such that (see (12) and (11))

$$(58) \quad B(x_R, D^{-2}d(R)) \cap E \subset R.$$

Then (54) holds for Q and any $R \in \min(\mathcal{S})$ by (56) and (52) provided $2c_1 \leq \alpha$. Since by (58) and (10) $d(x_{R_1}, x_{R_2}) \geq D^{-2}d(R_1) \geq D^{-2}h_{\mathcal{S}_0}(x_{R_1})$ for any distinct $R_1, R_2 \in \min(\mathcal{S})$, one has by (57) and Lemma 4.3

$$|\phi_Q(P_{W_Q}(x_{R_1})) - \phi_Q(P_{W_Q}(x_{R_2}))| \geq \frac{c_1^{\ell(Q)} d(x_{R_1}, x_{R_2})}{4D(1+2\delta)}$$

for any $R_1, R_2 \in \min(\mathcal{S})$. Thus by using (58), (14) and (53) and choosing c_1 small enough

$$\begin{aligned} d(t(R_1), t(R_2)) &\geq \frac{c_1^{\ell(Q)} d(x_{R_1}, x_{R_2})}{8D(1+2\delta)} - \frac{\sqrt{k}(l(t(R_1)) + l(t(R_2)))}{2} + \frac{c_1^{\ell(Q)} d(R_1, R_2)}{8D(1+2\delta)} \\ (59) \quad &\geq \left(\frac{1}{8D^3(1+2\delta)} - \sqrt{k}c_1D \right) c_1^{\ell(Q)} \max_{i=1,2} d(R_i) + \frac{c_1^{\ell(Q)} d(R_1, R_2)}{8D(1+2\delta)} \\ &\geq c_1^{\ell(Q)+1} (d(R_1) + d(R_2) + d(R_1, R_2)) \end{aligned}$$

for any distinct $R_1, R_2 \in \min(\mathcal{S})$. By (57) and the 1-Lipschitzness of P_{W_Q} also

$$(60) \quad d(t(R_1), t(R_2)) \leq \frac{c_1^{\ell(Q)} d(x_{R_1}, x_{R_2})}{4D} \leq d(R_1) + d(R_2) + d(R_1, R_2)$$

for any $R_1, R_2 \in \min(\mathcal{S})$.

For any $x \in F$ let $Q(x)$ be the smallest cube in \mathcal{T} which contains x . Then $Q(x) \in \mathcal{T}_2$ and one can define $f : F \rightarrow \mathbb{R}^k$ by setting

$$f(x) = \phi_{Q(x)}(P_{W_{Q(x)}}(x))$$

for all $x \in F$. From this on let $x, y \in F$ be distinct. Suppose first that $x, y \in Q$ for some $Q \in \mathcal{T}_2$. Let $Q = Q(\mathcal{S})$ be the smallest such cube.

Assume very first that there are distinct $R_1, R_2 \in \min(\mathcal{S})$ with $x \in R_1$ and $y \in R_2$. Since $x, y \notin F_1$ one has by (11)

$$(61) \quad d(x, y) \geq \frac{\tau(d(R_1) + d(R_2))}{3D} + \frac{d(R_1, R_2)}{3}.$$

Since $f(x) \in t(R_1)$ and $f(y) \in t(R_2)$ by definition of f , (56) and (54), one gets by using (53), (14), (60) and (61)

$$\begin{aligned} |f(x) - f(y)| &\leq d(t(R_1), t(R_2)) + d(t(R_1), t(R_2)) \\ &\leq D\sqrt{k}c_1 (d(R_1) + d(R_2)) + d(R_1) + d(R_2) + d(R_1, R_2) \\ &\leq Cd(x, y). \end{aligned}$$

By (59) one also has

$$|f(x) - f(y)| \geq d(t(R_1), t(R_2)) \geq c_1^{\ell(Q)+1} (d(R_1) + d(R_2) + d(R_1, R_2)) \geq c_1^M d(x, y).$$

Assume now that $y \in R_2 \in \min(\mathcal{S})$ and $x \notin R$ for all $R \in \min(\mathcal{S})$. Since now $f(x) = \phi_Q(P_{W_Q}(x))$, the argument used to establish (59) and (60) also gives

$$c_1^{\ell(Q)+1} (d(R_2) + d(x, R_2)) \leq d(f(x), t(R_2)) \leq d(R_2) + d(x, R_2).$$

Since further

$$\frac{\tau d(R_2)}{2D} + \frac{d(x, R_2)}{2} \leq d(x, y) \leq d(R_2) + d(x, R_2),$$

and $f(y) \in t(R_2)$, one has

$$c_1^M d(x, y) \leq |f(x) - f(y)| \leq Cd(x, y).$$

If $x \notin R$ and $y \notin R$ for all $R \in \min(\mathcal{S})$, then $f(x) = \phi_Q(P_{W_Q}(x))$ and $f(y) = \phi_Q(P_{W_Q}(y))$. In this case (57), Lemma 4.3 and the 1-Lipschitzness of P_{W_Q} give directly

$$\frac{c_1^M}{4D(1+2\delta)}d(x, y) \leq |f(x) - f(y)| \leq \frac{1}{4D}d(x, y).$$

Let now Q_1 be the largest cube in $\tilde{\Delta}$ which contains x but not y , and denote $Q_0 = O(Q_1)$.

Assume that $x, y \in R \in \min(\mathcal{S})$ (and that $Q(\mathcal{S})$ is still the smallest cube in \mathcal{T}_2 with $x, y \in Q$). Then necessarily $Q_0 = R \in \mathcal{T}_3$ or $Q_0 \in \mathcal{T}_1$, because otherwise $x, y \in Q_0 \notin \mathcal{S} \cup \mathcal{T}_1$ which contradicts the minimality of $Q(\mathcal{S})$. Now $y \in Q_2$ for some $Q_2 \in \mathcal{C}(Q_0) \setminus \{Q_1\}$ and $Q_1, Q_2 \in \mathcal{T}$. As before, by (11) and the definition of F

$$D^{-1}\tau d(Q_1) \leq d(x, y) \leq d(Q_0).$$

Further $f(x) \in t(Q_1) \subset t(Q_0)$ and $f(y) \in t(Q_2) \subset t(Q_0)$ (by definition of f , (56) and (54)). Thus by (52) (and (53)) and (14)

$$(62) \quad |f(x) - f(y)| \leq d(t(Q_0)) \leq c_1^{\ell(Q_0)}\sqrt{k}\alpha^{j_{Q_0}} \leq \sqrt{k}D^2\alpha\tau^{-1}d(x, y),$$

and by (55) and (11)

$$(63) \quad |f(x) - f(y)| \geq c_1^{\ell(Q_0)+1}\alpha^{j_{Q_0}} \geq c_1^M D^{-1}d(Q_0) \geq c_1^M D^{-1}d(x, y).$$

Finally assume that there does not exist $Q \in \mathcal{T}_2$ with $x, y \in Q$. Then $Q_1 \in \mathcal{R}_0$ or $Q_0 \in \mathcal{T}_1$. In the latter case (62) and (63) are obtained as above. In the former case

$$D^{-2}\tau\alpha^{m_0} \leq D^{-1}\tau d(Q_1) \leq d(x, y) \leq 2r \leq 2D\alpha^{m_0}$$

by (14) and the definition on F . Further

$$\alpha^{m_0} \leq |f(x) - f(y)| \leq (C+2)\alpha^{m_0}$$

by (51) and (52). (This is again because $f(x) \in t(Q_1)$ and $f(y) \in t(Q_2)$, where $Q_2 \in \mathcal{R}_0$ is such that $y \in Q_2$.) This gives

$$\frac{1}{2D}d(x, y) \leq |f(x) - f(y)| \leq \frac{D^2(C+2)}{\tau}d(x, y).$$

□

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